MASTERS IN ACTUARIAL SCIENCE

Risk Models

6/02/2020

1st part of the exam

Time allowed: 2 hours

Instructions:

1. This paper contains 8 questions and comprises 3 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all questions.
6. Begin your answer to each of the questions on a new page.
7. Marks are shown in brackets. Total marks: 140.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed formulary.

1. **[10]** Health insurers and the government are both putting pressure on hospitals to shorten the average length of stay (LOS) of their patients. A random sample of 20 hospitals had a mean LOS for women of 3.8 days and a standard deviation of 1.2 days. Assuming that the LOS is normally distributed, compute a 90% confidence interval to estimate the population expected LOS for women.

2. **[10]** Let $(X_1, X_2, \cdots, X_n)$ be a random sample from a Poisson population with unknown parameter λ. The focus is to estimate λ^2 using $T = \frac{\sum_{i=1}^{n} X_i (X_i - 1)}{n}$. Show that the estimator is unbiased.

3. **[15]** You are given the following random sample of 6 observations from the distribution of the random variable X: 2 4 4 5 7 10. Kernel smoothing is applied to estimate the density function of X. The kernel function used for the data point y is the normal distribution with mean y and variance 1. Use kernel smoothing to estimate $f(3)$ and $F(3)$.

4. **[10]** You are given the following sample (530, 6, 70, 64, 15, 82, 380, 96) from a population with density function $f(x|\theta) = \frac{20\theta x^{\theta-1}}{(x+20)^{\theta+1}}$. Obtain the maximum likelihood estimate for θ.

5. The following is a sample of 10 **payments**

   4 4 5+ 5+ 5+ 8 10+ 10+ 12 15

   where "+" indicates that the loss exceeded the policy limit. We also know that an ordinary deductible of 5 is in force for all the policies (i.e. claims below the deductible are not reported and claims above the deductible are paid on excess of the deductible). Assume that the claim amounts (not the payments) follow an exponential distribution with mean θ

   a) **[10]** Get the maximum likelihood estimate for θ.

   b) **[15]** Compute a maximum likelihood estimate for the probability that a claim is not reported. Using the asymptotic distribution of maximum likelihood estimators present a 95% confidence interval for this probability. Why is the confidence interval so wide?

6. **[15]** A **single loss**, x, is observed from a Pareto population with parameters $\alpha = 2$ and θ. In a Bayesian framework, assume that the improper prior $\pi(\theta) = \theta^{-2}, \quad \theta > 0$ has been defined. Compute the posterior distribution for θ and obtain Bayes estimate for θ against an absolute loss function.

7. You are given:

- Losses, X , follow an exponential distribution with mean θ .
- A random sample of 100 loss amounts

Loss Range	$(0;100]$	$(100;250]$	$(250;500]$	$(500;750]$	More than 750
Nº of losses	47	30	16	5	2

a) **[10]** Assuming a non-parametric framework, estimate $S(200)$ and also compute an estimate for $P(X > 200 \mid X < 500)$

b) **[5]** Write the log-likelihood function that is needed to estimate θ (just write the function, no computation is needed).

c) **[5]** Using R (function nlm) an actuary decided to minimize the function $-\ell(\theta)$. Assuming that this function has already been defined, you are given the following piece of output:

```
> theta.start=100
> out=nlm(minusloglik,theta.start,hessian=T)
>out
  $minimum
  [1] 124.2714
  $estimate
  [1] 174.2379
  $gradient
  [1] -1.059466e-07
  $hessian
        [,1]
  [1,] 0.003018865
  $code
  [1] 1
  $iterations
  [1] 8
```

Obtain a 95% approximate confidence interval for θ .

d) **[15]** Using the chi-square goodness of fit test, test the adequacy of an exponential distribution with mean 200 to the observed sample.

8. Let us consider a random sample with size 11 from a Weibull population with parameters $\tau = 3$ and $\theta = 100$.

a) **[5]** Write the density function of the sample median.

b) **[15]** Explain how to use simulation to get an approximate value for the probability that the sample median is smaller than the sample mean.

Blank page

SOLUTIONS

1.

Pivotal Quantity: $T = \dfrac{\bar{X} - \mu}{S / \sqrt{20}} \sim t(19)$ $\qquad t_{0.05} = 1.729$

The confidence interval is then given by $\bar{x} \pm t_{0.05} s / \sqrt{19}$, i.e. $3.8 \pm 1.729 \times 1.2 / \sqrt{20}$

90% CI (3.336; 4.264)

2.

The estimator is unbiased if its expected value is equal to λ^2

$$E(T) = E\left(\frac{\sum_{i=1}^{n} X_i (X_i - 1)}{n}\right) = E\left(\frac{\sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} X_i}{n}\right) = E\left(\frac{\sum_{i=1}^{n} X_i^2}{n}\right) - E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right).$$

$$= \frac{\sum_{i=1}^{n} E(X_i^2)}{n} - E\left(\bar{X}\right) = \frac{\sum_{i=1}^{n}\left(\operatorname{var}(X_i) + \left(E(X_i)\right)^2\right)}{n} - \lambda = \frac{\sum_{i=1}^{n}\left(\lambda + \lambda^2\right)}{n} - \lambda = \lambda^2$$

3.

$$k_y(3) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(3-y)^2}{2}}$$

$$\hat{f}(3) = \frac{1}{6}k_2(3) + \frac{2}{6}k_4(3) + \frac{1}{6}k_5(3) + \frac{1}{6}k_7(3) + \frac{1}{6}k_{10}(3)$$
$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{6}e^{-\frac{(3-2)^2}{2}} + \frac{2}{6}e^{-\frac{(3-4)^2}{2}} + \frac{1}{6}e^{-\frac{(3-5)^2}{2}} + \frac{1}{6}e^{-\frac{(3-7)^2}{2}} + \frac{1}{6}e^{-\frac{(3-10)^2}{2}}\right)$$
$$= \frac{1}{\sqrt{2\pi}}\left(\frac{3}{6}e^{-0.5} + \frac{1}{6}e^{-2} + \frac{1}{6}e^{-8} + \frac{1}{6}e^{-24.5}\right) = 0.13$$

$$K_y(3) = \Phi\left(\frac{3-y}{1}\right) = \Phi\left(3-y\right)$$

$$\hat{F}(3) = \frac{1}{6}K_2(3) + \frac{2}{6}K_4(3) + \frac{1}{6}K_5(3) + \frac{1}{6}K_7(3) + \frac{1}{6}K_{10}(3)$$
$$= \frac{1}{6}\Phi(1) + \frac{2}{6}\Phi(-1) + \frac{1}{6}\Phi(-2) + \frac{1}{6}\Phi(-4) + \frac{1}{6}\Phi(-7)$$
$$= \frac{1}{6}\times 0.84134 + \frac{2}{6}\times 0.15866 + \frac{1}{6}\times 0.02275 + \frac{1}{6}\times 0.00003 + \frac{1}{6}\times 0.00000 = 0.196907$$

4.

$$\ell(\theta)=\sum_{i=1}^{n}\left(\ln 20+\ln\theta+(\theta-1)\ln x_i-(\theta+1)\ln(x_i+20)\right)$$

$$\ell'(\theta)=\sum_{i=1}^{n}\left(\frac{1}{\theta}+\ln x_i-\ln(x_i+20)\right)=\frac{n}{\theta}+\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\ln(x_i+20)$$

$$\ell'(\theta)=0\Leftrightarrow\frac{n}{\theta}=\sum_{i=1}^{n}\ln(x_i+20)-\sum_{i=1}^{n}\ln x_i\Leftrightarrow\theta=\frac{n}{\sum_{i=1}^{n}\ln(x_i+20)-\sum_{i=1}^{n}\ln x_i}=\frac{n}{\sum_{i=1}^{n}\ln\left(\frac{x_i+20}{x_i}\right)}$$

$$\ell''(\theta)=\sum_{i=1}^{n}\left(-\frac{1}{\theta^2}\right)=-\frac{n}{\theta^2}<0$$

And then the maximum likelihood estimate of θ is $\hat{\theta}=\frac{n}{\sum_{i=1}^{n}\ln\left(\frac{x_i+20}{x_i}\right)}=2.400447$

5.

Let x_i be the claim amount and y_i the payment of claim i. The observed claim amounts are then : 9 9 10+ 10+ 10+ 13 15+ 15+ 17 20.

The likelihood is

$$L(\theta)=\left(\frac{f(9|\theta)}{S(5|\theta)}\right)^2\times\left(\frac{S(10|\theta)}{S(5|\theta)}\right)^3\times\frac{f(13|\theta)}{S(5|\theta)}\times\left(\frac{S(15|\theta)}{S(5|\theta)}\right)^2\times\frac{f(17|\theta)}{S(5|\theta)}\times\frac{f(20|\theta)}{S(5|\theta)}$$

And the log-likelihood

$$\ell(\theta)=2\ln f(9|\theta)+3\ln S(10|\theta)+\ln f(13|\theta)+2\ln S(15|\theta)+\ln f(17|\theta)+\ln f(20|\theta)-10\ln S(5|\theta)$$

$$=-5\ln\theta-\frac{18}{\theta}-\frac{30}{\theta}-\frac{13}{\theta}-\frac{30}{\theta}-\frac{17}{\theta}-\frac{20}{\theta}+\frac{50}{\theta}=-5\ln\theta-\frac{78}{\theta}$$

$$\ell'(\theta)=-\frac{5}{\theta}+\frac{78}{\theta^2}\qquad\ell'(\theta)=0\Leftrightarrow\frac{5}{\theta}=\frac{78}{\theta^2}\Leftrightarrow\theta=\frac{78}{5}=15.6$$

$$\ell''(\theta)=\frac{5}{\theta^2}-\frac{156}{\theta^3}\qquad\ell''(\theta)\big|_{\theta=15.6}=\frac{5}{\theta^2}-\frac{156}{\theta^3}=-0.02055<0$$ and then $\hat{\theta}=15.6$

b.

A claim is not reported if its amount is smaller than 5 (the deductible) and $P(X<5)=1-e^{-5/\theta}$

Then $\hat{P}(X<5)=1-e^{-5/15.6}=0.2742$.

$$g(\theta)=1-e^{-5/\theta}\qquad g'(\theta)=\frac{5}{\theta^2}e^{-5/\theta}\qquad \hat{\text{var}}(\hat{\theta})\approx\frac{1}{0.2055}=48.672$$

$$\hat{\text{var}}\left(\hat{P}(X<5)\right)=\left(\frac{5}{\hat{\theta}^2}e^{-5/\hat{\theta}}\right)^2 48.672=0.01082$$

The 95% CI is then $\left(0.2742-1.96\sqrt{0.01082};0.2742+1.96\sqrt{0.01082}\right)$ i.e (0.0703; 0.4781)

The width of the interval is so large because the sample size is very small and furthermore we face censoring and truncation problems.

6.

$\pi(\theta)=\theta^{-2}$, $\theta>0$ $\qquad\qquad L(\theta|x) = f(x|\theta) = \frac{2\,\theta^2}{(x+\theta)^3} \quad \theta > 0$

$\pi_{\theta|x}(\theta) \propto \theta^{-2}\times\frac{\theta^2}{(x+\theta)^3}=(x+\theta)^{-3}$, $\theta>0$ core of a Pareto distribution with parameters 2 and x or we can compute $\int_0^\infty (x+\theta)^{-3}d\theta=\left[\frac{(x+\theta)^{-2}}{-2}\right]_0^\infty=\frac{1}{2x^2}$

$\pi_{\theta|x}(\theta)=2x^2(x+\theta)^{-3}$ Pareto distribution with parameters 2 and x

Bayes estimate against an absolute loss function: $median(\pi_{\theta|x}(\theta))=x\left(\sqrt{2}-1\right)\approx 0.4142\,x$

As $1-\left(\frac{x}{\theta+x}\right)^2=\frac{1}{2}\Leftrightarrow\frac{1}{2}=\left(\frac{x}{\theta+x}\right)^2\Leftrightarrow\frac{1}{\sqrt{2}}=\left(\frac{x}{\theta+x}\right)\Leftrightarrow\theta+x=x\sqrt{2}\Leftrightarrow\theta=x\left(\sqrt{2}-1\right)$

7.

a)

$$S_n(200)=1-\frac{250\times47/100-100\times77/100}{250-100}-\frac{77/100-47/100}{250-100}\times200$$
$$=1-\frac{40.5}{150}-\frac{60}{150}=\frac{49.5}{150}=0.33$$

$$P(X>200\,|\,X<500)=\frac{P(200<X<500)}{P(X<500)}=\frac{F(500)-F(200)}{F(500)}=1-\frac{F(200)}{F(500)}$$

As $F_n(200)=1-0.33=0.67$ and $F_n(500)=93/100=0.93$ we obtain

$$\tilde{P}(X>200\,|\,X<500)==1-\frac{F_n(200)}{F_n(500)}=1-\frac{0.67}{0.93}=0.2796$$

b)

$$L(\theta)=\left(F(100\,|\,\theta)-0\right)^{47}\times\left(F(250\,|\,\theta)-F(100\,|\,\theta)\right)^{30}\times\left(F(500\,|\,\theta)-F(250\,|\,\theta)\right)^{16}$$
$$\times\left(F(750\,|\,\theta)-F(500\,|\,\theta)\right)^{5}\times\left(1-F(750\,|\,\theta)\right)^{2}$$
$$=\left(1-\mathrm{e}^{-100/\theta}\right)^{47}\times\left(\mathrm{e}^{-100/\theta}-\mathrm{e}^{-250/\theta}\right)^{30}\times\left(\mathrm{e}^{-250/\theta}-\mathrm{e}^{-500/\theta}\right)^{16}\times\left(\mathrm{e}^{-500/\theta}-\mathrm{e}^{-750/\theta}\right)^{5}\times\mathrm{e}^{-1500/\theta}$$

$$\ell(\theta)=47\ln\left(1-\mathrm{e}^{-100/\theta}\right)+30\ln\left(\mathrm{e}^{-100/\theta}-\mathrm{e}^{-250/\theta}\right)+16\ln\left(\mathrm{e}^{-250/\theta}-\mathrm{e}^{-500/\theta}\right)+5\ln\left(\mathrm{e}^{-500/\theta}-\mathrm{e}^{-750/\theta}\right)-\frac{1500}{\theta}$$

c)

From R output we got $\hat{\theta}=174.2379$ and $\hat{\text{var}}(\hat{\theta})=1/0.003018865=331.25$ and then a 95% CI for θ is given by $\hat{\theta}\pm 1.96\sqrt{\hat{\text{var}}(\hat{\theta})}$, i.e. (138.6; 209.9)

d)

$H_0: X \sim \text{Exponential}(200) \qquad H_1: H_0 \text{ false}$

Test statistic: $Q=\sum_{j=1}^{5}\frac{(O_j-E_j)^2}{E_j}\overset{a}{\sim}\chi^2_{(4)}$

Loss Range	Nº of losses	Prob	Expected	Chi squared
$(0;100]$	47	0.3935	39.35	1.4885
$(100;250]$	30	0.3200	32.00	0.1253
$(250;500]$	16	0.2044	20.44	0.9652
$(500;750]$	5	0.0585	5.85	0.1253
More than 750	2	0.0235	2.35	0.0526
Total	**100**			**2.7570**

p.value=$P(Q\geq 2.757)=0.5993$ Do not reject the null, i.e. the exponential with mean 200 is an acceptable distribution for the population.

8.

a)

Let X represents the population and M be the sample median.

As we know $f_X(x)=\tau\, x^{\tau-1}\,\theta^{-\tau}\,e^{-(x/\theta)^\tau}$ and $F_X(x)=1-e^{-(x/\theta)^\tau}$.

As $f_M(m)=\frac{11!}{5!1!5!}\left(F_X(m)\right)^5 f_X(m)\left(1-F_X(m)\right)^5$ we get

$$f_M(m)=\frac{11!}{(5!)^2}\left(1-e^{-(m/\theta)^\tau}\right)^5 \tau\, m^{\tau-1}\,\theta^{-\tau}\,e^{-(m/\theta)^\tau}\left(e^{-(m/\theta)^\tau}\right)^5=\frac{11!}{(5!)^2}\tau\,\theta^{-\tau}m^{\tau-1}\left(1-e^{-(m/\theta)^\tau}\right)^5 e^{-6(m/\theta)^\tau}$$

$$=\frac{11!}{(5!)^2}\left(3\times 100^{-3}\right)m^2\left(1-e^{-(m/100)^3}\right)^5 e^{-6(m/100)^3}$$

$$=0.00816\, m^2\left(1-e^{-(m/100)^3}\right)^5 e^{-6(m/100)^3}$$

b)

Define the number of replicas NR to be used

Define 2 arrays, $s.avg$ and $s.med$ with NR elements each

For each replica, $j = 1, 2, \cdots, NR$

- Generate the sample
  - Generate 11 pseudo random numbers
  - Compute $x_i = \theta\left(-\ln(1-u)\right)^{1/\tau}$
- Compute the sample mean $\bar{x} = \dfrac{\sum_{i=1}^{11} x_i}{11}$ and the sample median m (sort the 11 values and choose the middle one) and keep these values, $s.avg_j = \bar{x}$ and $s.med_j = m$

The approximate probability will be given by $\dfrac{\#\left\{s.med_j < s.avg_j\right\}}{NR}$

Blank page

MASTERS IN ACTUARIAL SCIENCE

Risk Models

6/02/2020

2nd part of the exam

Time allowed: 1 hour

Instructions:

1. This paper contains 2 questions and comprises 2 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes reading time. You must not start writing your answers until instructed to do so.
4. You are requested to summarize your answers on exam sheets. You can add any comments you think necessary to understand your answers.
5. At the end of the exam, you should submit your R files to Aquila using your usual username and password.
6. Attempt all questions.
7. Marks are shown in brackets. Total marks: 60.
8. The distributed formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed formulary.

1. File glass1.csv presents 10 characteristics for 214 glass samples as well as the type of glass (cod.type and type columns). The characteristics are:

   1. Id number: 1 to 214
   2. RI: refractive index
   3. Na: Sodium (unit measurement: weight percent in corresponding oxide, as are attributes 4-10)
   4. Mg: Magnesium
   5. Al: Aluminum
   6. Si: Silicon
   7. K: Potassium
   8. Ca: Calcium
   9. Ba: Barium
   10. Fe: Iron

   a. **[5]** Test if the correlation coefficient between RI and Ca is equal to zero against the alternative that it is negative.
   b. **[10]** Using a Kolmogorov-Smirnov test, test if it is acceptable to consider that the variable Na follows a normal distribution with mean 13 and standard deviation equal to 1.
   c. **[15]** Using a PCA answer to the following questions:
      i. How many principal components should be retained?
      ii. Present the loadings for the retained components. Identify the variables that are more closely related to each retained component
      iii. What are the coordinates of the first three observations according to the retained components?

2. File Pareto.csv presents 200 observations from a population that follows a Pareto distribution with parameters α and θ.
   a. **[15]** Obtain a maximum likelihood estimate for the unknown parameters and also obtain $\hat{\text{cov}}(\hat{\alpha},\hat{\theta})$.
   b. **[15]** Using the likelihood ratio test, test $H_0:\alpha=6$ against $H_1:\alpha\neq 6$

Solution

1.

a)

$H_0 : \rho = 0$ $\qquad$ $H_1 : \rho < 0$

Command: cor.test(RI,Ca,alternative="less")

```
        Pearson's product-moment correlation

data:  RI and Ca
t = 20.14, df = 212, p-value = 1
alternative hypothesis: true correlation is less than 0
95 percent confidence interval:
 -1.0000000  0.8458651
sample estimates:
      cor
0.8104027
```

As p-value is close to 1 we do not reject the null, i.e. the correlation coefficient is negative

b)

$H_0 : X \sim n(13,1)$ $\qquad$ $H_1 : H_0$ is false

Command: ks.test(Na,"pnorm",mean=13,sd=1)

```
     One-sample Kolmogorov-Smirnov test

data:  Na
D = 0.27998, p-value = 5.329e-15
alternative hypothesis: two-sided

Warning message:
In ks.test(Na, "pnorm", mean = 13, sd = 1) :
  ties should not be present for the Kolmogorov-Smirnov test
```

As p-value is very close to 0 we strongly reject the null, i.e. the population's distribution cannot be considered as a normal (13,1)

c)

Commands:

```
x=cbind(RI,Na,Mg,Al,Si,K,Ca,Ba,Fe)
out.PCA=prcomp(x,scale=T); out.PCA
```

Standard deviations (1, .., p=9):
[1] 1.58466518 1.43180731 1.18526115 1.07604017 0.95603465 0.72638502 0.60741950
[8] 0.25269141 0.04011007

Rotation (n x k) = (9 x 9):

	PC1	PC2	PC3	PC4	PC5	PC6
RI	0.5451766	-0.28568318	0.0869108293	0.14738099	-0.073542700	0.11528772
Na	-0.2581256	-0.27035007	-0.3849196197	0.49124204	0.153683304	-0.55811757
Mg	0.1108810	0.59355826	0.0084179590	0.37878577	0.123509124	0.30818598
Al	-0.4287086	-0.29521154	0.3292371183	-0.13750592	0.014108879	-0.01885731
Si	-0.2288364	0.15509891	-0.4587088382	-0.65253771	0.008500117	0.08609797
K	-0.2193440	0.15397013	0.6625741197	-0.03853544	-0.307039842	-0.24363237
Ca	0.4923061	-0.34537980	-0.0009847321	-0.27644322	-0.188187742	-0.14866937
Ba	-0.2503751	-0.48470218	0.0740547309	0.13317545	0.251334261	0.65721884
Fe	0.1858415	0.06203879	0.2844505524	-0.23049202	0.873264047	-0.24304431

	PC7	PC8	PC9
RI	0.08186724	0.75221590	0.02573194
Na	0.14858006	0.12769315	-0.31193718
Mg	-0.20604537	0.07689061	-0.57727335
Al	-0.69923557	0.27444105	-0.19222686
Si	0.21606658	0.37992298	-0.29807321
K	0.50412141	0.10981168	-0.26050863
Ca	-0.09913463	-0.39870468	-0.57932321
Ba	0.35178255	-0.14493235	-0.19822820
Fe	0.07372136	0.01627141	-0.01466944

Following Kaiser's criterion, we should retain 4 components.

The loadings are given by (command: cor(x,out.PCA$x[,1:4])):

	PC1	PC2	PC3	PC4
RI	**0.8639224**	-0.4090433	0.103012029	0.15858786
Na	-0.4090426	-0.3870892	-0.456230271	**0.52859617**
Mg	0.1757092	**0.8498611**	0.009977480	0.40758870
Al	-0.6793596	-0.4226860	0.390231965	-0.14796189
Si	-0.3626290	0.2220718	**-0.543689765**	**-0.70215679**
K	-0.3475869	0.2204556	**0.785323363**	-0.04146568
Ca	**0.7801403**	-0.4945173	-0.001167165	-0.29746401
Ba	-0.3967607	**-0.6940001**	0.087774195	0.14330214
Fe	0.2944966	0.0888276	0.337148189	-0.24801867

The first component is linked to RI and Ca
The second to Mg and Ba
The third to K (and Si)
The fourth to Na and Si

The coordinates are the following (command: out.PCA$x[1:3,1:4])

	PC1	PC2	PC3	PC4
[1,]	0.21347827	1.0625603	-0.20236555	-0.2926908
[2,]	-0.07073491	1.2981588	0.21541867	-0.6338652
[3,]	-0.15540960	0.7364229	-0.02881056	0.2724099

2.

a)

Commands:

```
dta=read.csv("F:/PCA dataSet/ML.Pareto.csv")
x=dta$x
minusloglik.Pareto=function(param,x){
  alpha=param[1]; theta=param[2]
  return(-sum(log(alpha)+alpha*log(theta)-(alpha+1)*log(x+theta)))
 }

param.start=c(2,100)
out.Pareto=nlm(minusloglik.Pareto,param.start,hessian=T,x=x)
            Warning messages:
            1: In log(alpha) : NaNs produced
            2: In nlm(minusloglik.Pareto, param.start, hessian = T, x = x) :
              NA/Inf replaced by maximum positive value
            3: In log(alpha) : NaNs produced
            4: In nlm(minusloglik.Pareto, param.start, hessian = T, x = x) :
              NA/Inf replaced by maximum positive value
> out.Pareto
$minimum
[1] 1148.871
$estimate
[1]   3.652143 319.212106
$gradient
[1]  7.470913e-06 -1.018584e-07
$hessian
         [,1]       [,2]
[1,] 14.9915892 -0.13466727
[2,] -0.1346673  0.00126104
$code
[1] 1
$iterations
[1] 26
```

Then the maximum likelihood estimates are $\hat{\alpha} = 3.652143$ and $\hat{\theta} = 319.212106$.

To get an estimate of $\text{cov}(\hat{\alpha}, \hat{\theta})$ we must first compute the inverse of the hessian matrix (remember that it is the hessian matrix for minus the loglikelihood) and then use the out diagonal element, $\hat{\text{cov}}(\hat{\alpha}, \hat{\theta}) = 174.9573$.

```
solve(out.Pareto$hessian)
        [,1]      [,2]
[1,]   1.63832   174.9573
[2,] 174.95727 19476.8001
```

b)

$H_0 : \alpha = 6 \qquad H_1 : \alpha \neq 6$

The loglikelihood of the unconstrained model at the maximum likelihood estimates is (see previous question): $\ell(\hat{\alpha}, \hat{\theta}) = -1148.871$

Now we need to get the maximum likelihood estimate for the constrained model:

Commands:

```
minusloglik.Pareto1=function(theta,x,alpha){
  return(-sum(log(alpha)+alpha*log(theta)-(alpha+1)*log(x+theta)))
 }
theta.start=theta.hat
out.Pareto1=nlm(minusloglik.Pareto1,theta.start,hessian=T,x=x,alpha=6)
out.Pareto1
$minimum
[1] 1149.632
$estimate
[1] 577.0659
$gradient
[1] -3.152135e-09
$hessian
          [,1]
[1,] 0.0004415654
$code
[1] 1
$iterations
[1] 8
```

Then the loglikelihood of the constrained model at the maximum likelihood estimate is $\ell_0(\hat{\theta}^*) = -1149.632$

Now , compute the test statistic and its p-value

```
> test.statistic=-2*(-out.Pareto1$minimum+out.Pareto$minimum); test.statistic
[1] 1.522014
> p.value=pchisq(test.statistic,1,lower=F); p.value
[1] 0.217315
```

As the p-value is much greater than the usual levels of significance we do not reject the null, i.e. it is acceptable to consider that $\alpha = 6$